

IDEALS GENERATED BY DIAGONAL 2-MINORS

VIVIANA ENE AND AYESHA ASLOOB QURESHI

ABSTRACT. With a simple graph G on $[n]$, we associate a binomial ideal P_G generated by diagonal minors of an $n \times n$ matrix $X = (x_{ij})$ of variables. We show that for any graph G , P_G is a prime complete intersection ideal and determine the divisor class group of $K[X]/P_G$. By using these ideals, one may find a normal domain with free divisor class group of any given rank.

INTRODUCTION

Classically, with a simple graph G on the vertex set $[n]$, one associates the so-called edge ideal $I(G)$ in the polynomial ring $K[x_1, \dots, x_n]$ over a field K . Recently, binomial edge ideals have been considered in [7] and, independently, in [9]. The binomial edge ideal J_G of G is generated by the binomials $x_i y_j - x_j y_i \in K[x_1, \dots, x_n, y_1, \dots, y_n]$, where $\{i, j\}$ is an edge of G . For instance, the ideal of all 2-minors of a $2 \times n$ matrix of variables is a special example of a binomial edge ideal.

The study of the ideals of 2-minors of matrices of variables is motivated by their relevance in algebraic statistics and other fields as it was shown in [4]. For more recent results on ideals generated by 2-minors one may consult [8], [6], [10].

In this paper, we introduce a new class of ideals of 2-minors associated with graphs. Let $X = (x_{ij})$ be an $n \times n$ -matrix of variables and $S = K[X]$ the polynomial ring over a field K in the variables $\{x_{ij}\}_{1 \leq i, j \leq n}$. Let G be a simple graph on the vertex set $[n]$. With this graph we associate an ideal generated by diagonal 2-minors of X in the following way. For $1 \leq i < j \leq n$ we denote by f_{ij} the diagonal 2-minor of X given by the elements at the intersections of the rows i, j and the columns i, j , that is, $f_{ij} = x_{ii}x_{jj} - x_{ij}x_{ji}$. Let P_G be the ideal of S generated by the binomials f_{ij} where $\{i, j\}$ is an edge of G .

With respect to the lexicographical order on S induced by the natural order of variables, namely $x_{11} > x_{12} > \dots > x_{1n} > x_{21} > \dots > x_{nn}$, the reduced Gröbner basis consists of binomials of degree at most 4 which have squarefree initial monomials, as we show in Theorem 1.2. But if we consider the reverse lexicographical order induced by the natural order of the variables, then the generators of P_G form a Gröbner basis of P_G and moreover, they form even a regular sequence, therefore $\text{height}(P_G)$ equals the number of edges of G and $\text{in}_\prec(P_G)$ is squarefree. Here \prec denotes the reverse lexicographic order.

We show in Proposition 1.3 that P_G is a prime ideal, hence the ring $R_G = S/P_G$ is a normal domain.

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In the last section we study the divisor class group $\text{Cl}(R_G)$. We show in Theorem 2.2 that $\text{Cl}(R_G)$ is free and we express its rank in terms of the graph's data. Finally, in Proposition 2.3, we give sharp bounds for the possible rank of $\text{Cl}(R_G)$ when G has a given number of edges. Every abelian group is the class group of a Krull domain as shown by Claborn [1]. By using ideals generated by diagonal 2-minors, one may find an example of a normal domain with free divisor class group of any given rank.

1. GRÖBNER BASES OF IDEALS GENERATED BY DIAGONAL 2-MINORS

Let $X = (x_{ij})_{1 \leq i, j \leq n}$ be a square matrix of variables and $S = K[X]$ the polynomial ring over a field K . For any $1 \leq i < j \leq n$, let $f_{ij} = x_{ii}x_{jj} - x_{ij}x_{ji}$. Let G be a simple graph on the vertex set $[n]$ and $P_G = (f_{ij} : \{i, j\} \in E(G))$ where $E(G)$ is the set of edges of G . In this section we compute the Gröbner bases of P_G with respect to the lexicographic and the reverse lexicographic order on S .

Usually, when we study ideals generated by minors of matrices of variables, one uses the lexicographic order induced by the natural order of variables, namely, row by row from left to right. This monomial order selects as initial monomial of each 2-minor of X the product of variables of the main diagonal. It will turn out that this order gives a rather large Gröbner basis for P_G . But if we consider a monomial order on S which selects the product of the variables on the anti-diagonal of each 2-minor as initial monomial, then the generators of P_G form a Gröbner basis. More precisely, let us consider the reverse lexicographic order on the ring S . With respect to this order, $\text{in}_{\prec} f_{ij} = x_{ij}x_{ji}$, that is, the initial monomial comes from the anti-diagonal of the minor f_{ij} for any $1 \leq i < j \leq n$.

Proposition 1.1. *Let G be a simple graph on the vertex set $[n]$ with the edge set $E(G)$ and let $P_G = (f_{ij} : \{i, j\} \in E(G))$ be the binomial ideal generated by the diagonal 2-minors associated with G . Then the set of generators of P_G is the reduced Gröbner basis of P_G with respect to the reverse lexicographic order. Moreover, P_G is a complete intersection of $\text{height}(P_G) = |E(G)|$.*

Proof. All claims follow immediately if we notice that the initial monomials of f_{ij} with respect to \prec form a regular sequence. \square

We now consider the lexicographic order induced by the natural order of variables, namely,

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > x_{22} > \cdots > x_{2n} > \cdots > x_{n1} > x_{n2} > \cdots > x_{nn}.$$

With respect to this order, $\text{in}_{\prec} f_{ij} = x_{ii}x_{jj}$, in other words, the initial monomial comes from the main diagonal of the minor f_{ij} .

Theorem 1.2. *Let G be a simple graph on the set $[n]$ and let P_G be its associated ideal. The initial ideal of P_G with respect to the lexicographic order induced by the natural order of indeterminates is generated by squarefree monomials of degree at most 4.*

Proof. We compute a Gröbner basis \mathcal{G}_{lex} of P_G by applying Buchberger's criterion.

We first compute the S -polynomials of the generators of P_G . Let f_{ij} and f_{kl} be two binomials in the generating set P_G . We consider the non trivial case when $\gcd(\text{in}_<(f_{ij}), \text{in}_<(f_{kl})) \neq 1$. We may have one of the following possibilities:

- (i) $x_{jj} = x_{kk}$ (or $x_{ii} = x_{ll}$),
- (ii) $x_{ii} = x_{kk}$,
- (iii) $x_{jj} = x_{ll}$.

Consider the case (i) as shown in Figure 1. Then, the S -polynomial $S(f_{ij}, f_{jl}) = x_{ii}x_{jl}x_{lj} - x_{ll}x_{ij}x_{ji}$ must be added to \mathcal{G}_{lex} , since none of its monomials is divisible by a monomial of the form $x_{aa}x_{bb}$.

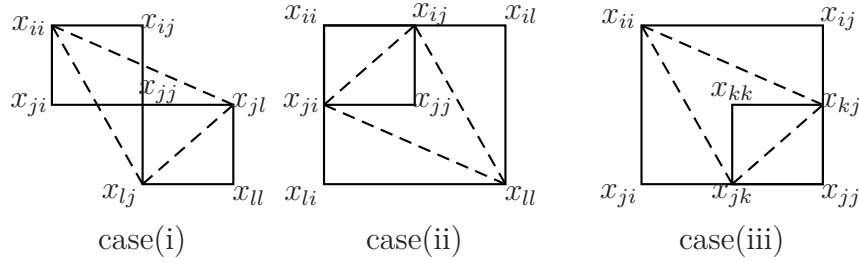


FIGURE 1.

For case (ii), see Figure 1, we may assume that, for instance, $j < l$, and we get $S(f_{ij}, f_{il}) = x_{jj}x_{il}x_{li} - x_{ll}x_{ij}x_{ji}$ that should also be added to \mathcal{G}_{lex} . Moreover, $\text{in}_<(S(f_{ij}, f_{il})) = x_{ll}x_{ij}x_{ji}$.

In case (iii), we may choose $i < k$, and get $S(f_{ij}, f_{kj}) = x_{ii}x_{jk}x_{kj} - x_{kk}x_{ji}x_{ij}$ in \mathcal{G}_{lex} , with $\text{in}_<(S(f_{ij}, f_{kj})) = x_{ii}x_{jk}x_{kj}$; see Figure 1.

By the above computation of S -polynomials, we have got all the binomials of degree 3 which belong to the Gröbner basis of P_G . Next we investigate the S -polynomials $S(g, f_{ij})$, where g is a binomial of degree 3 and f_{ij} is a quadratic binomial. We are going to discuss only those cases when the S -polynomial does not reduce to 0, and hence contributes to the Gröbner basis.

Case 1: Let $g = x_{ii}x_{jl}x_{lj} - x_{ll}x_{ij}x_{ji}$ with $i < j < l$ and $f_{iq} = x_{ii}x_{qq} - x_{qi}x_{iq}$ with $i < q$. We have $S(f_{iq}, g) = x_{jl}x_{lj}x_{qi}x_{iq} - x_{qq}x_{ll}x_{ij}x_{ji}$. If $\text{in}_<(S(g, f_{iq}))$ is the first monomial, that is, $q < j$, then we add $S(g, f_{iq})$ to \mathcal{G}_{lex} since $x_{jl}x_{lj}x_{qi}x_{iq}$ is not divisible by any of the previous initial monomials which we have obtained so far; see Figure 2.

Otherwise, that is, for $q > j$, we reduce $S(g, f_{iq})$ modulo a binomial of degree 3, namely the one which appears when we take $S(f_{ij}, f_{iq})$, and get

$$S(g, f_{iq}) = -x_{ll}S(f_{ij}, f_{iq}) - x_{iq}x_{qi}f_{jl}.$$

Therefore, $S(g, f_{iq})$ reduces to zero. Now it remains to consider $S(g, f_{qi})$, where $q < i$. By proceeding as before, it follows that $S(g, f_{qi})$ reduces to zero modulo a binomial of degree 3 and a binomial of degree 2.

Case 2: Let $g = x_{ll}x_{ij}x_{ji} - x_{jj}x_{il}x_{li}$ and $f_{lq} = x_{ll}x_{qq} - x_{ql}x_{lq}$, with $l < q$. Then we have $S(g, f_{lq}) = x_{ij}x_{ji}x_{lq}x_{ql} - x_{qq}x_{jj}x_{il}x_{li}$. Since $\text{in}_<(S(g, f_{lq}))$ is not divisible by

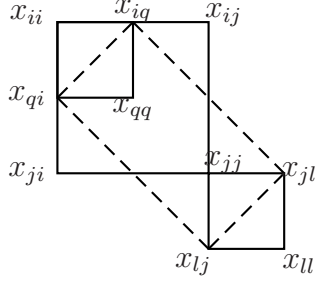


FIGURE 2.

any initial monomial obtained so far, we add this polynomial to our Gröbner basis. We get the same conclusion if we take $S(g, f_{ql})$ with $l > q$; see Figure 3.

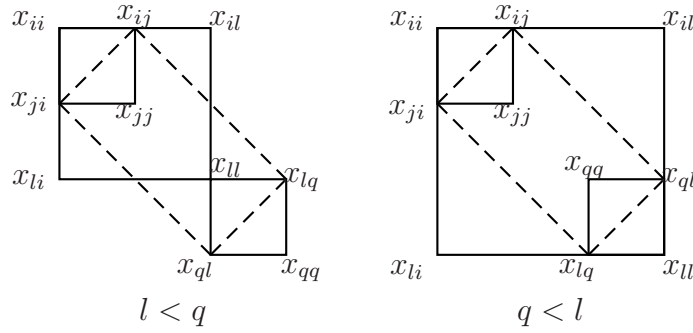


FIGURE 3.

Case 3: Let $g = x_{ii}x_{kj}x_{jk} - x_{ij}x_{ji}x_{kk}$ and $f_{iq} = x_{ii}x_{qq} - x_{iq}x_{qi}$ with $i < q$. Then, we have $S(g, f_{iq}) = x_{iq}x_{qi}x_{jk}x_{kj} - x_{ij}x_{ji}x_{kk}x_{qq}$. If $q < j$, then $\text{in}_<(S(g, f_{iq}))$ is a new initial monomial, hence we add $S(g, f_{iq})$ to our Gröbner basis; see Figure 4. Otherwise, that is, if $q > j$, then, as we did in Case 1, we observe that $S(g, f_{iq})$ reduces to zero modulo a binomial of degree 3 and a binomial of degree 2.

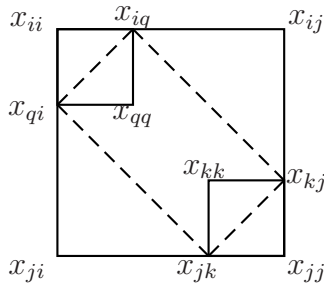


FIGURE 4.

If $q < i$, we may again reduce $S(g, f_{qi})$. So far, apart from the original generators of P_G , we have in the Gröbner basis with respect to the lexicographic order, binomials of degree 3 and 4.

Now we discuss the S -polynomial of degree 3 binomials. Let $g = x_{ii}x_{jl}x_{lj} - x_{ll}x_{ij}x_{ji}$ and $h = x_{pp}x_{rq}x_{qr} - x_{rr}x_{pq}x_{qp}$ be any two binomial of the form that we obtained by computing S -polynomials of the generators of P_G . If $\text{gcd}(\text{in}_<(g), \text{in}_<(h)) \neq 1$,

then we either have $x_{ii} = x_{pp}$ or $\{x_{jl}, x_{lj}\} = \{x_{qr}, x_{rq}\}$. In the first case, we obtain a binomial of degree 5 which reduces to 0, and in the second case, $S(f, g)$ is a binomial of degree 4 which is also reducible. To understand this, consider the following example. Let $f = x_{qp}x_{pq}x_{rr} - x_{qq}x_{rp}x_{pr}$ and $g = x_{ii}x_{jl}x_{lj} - x_{ll}x_{ij}x_{ji}$ where f and g are obtained as in case (i) and case(ii) of Figure 1, respectively.

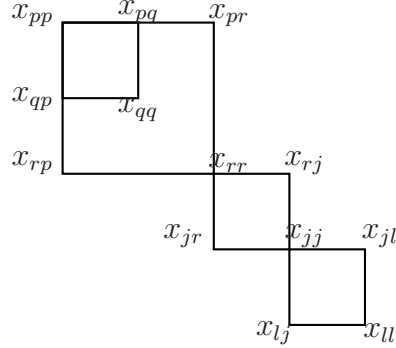


FIGURE 5.

If we let $x_{ii} = x_{rr}$, then $S(f, g)$ reduces to 0 with respect to $x_{qp}x_{pq}x_{jr}x_{rj} - x_{qq}x_{jj}x_{rp}x_{pr}$ and $x_{ll}x_{jj} - x_{jl}x_{lj}$; see Figure 5. If $x_{jl} = x_{pq}$ then $S(f, g)$ is the product of x_{qq} and $x_{ii}x_{rp}x_{pr} - x_{rr}x_{ip}x_{pi}$.

By a careful computation of S -polynomials in the cases when we consider S -polynomials of degree 4 binomials with binomials of degree 2, 3 and 4, we see that they reduce to 0 and hence \mathcal{G}_{lex} consists of binomials with squarefree initial term of degree at most 4. \square

Proposition 1.3. P_G is a prime ideal, thus S/P_G is a domain.

Proof. We may assume that G has no isolated vertices. Let $\{1, i\}$ be an edge of G and let G' be the subgraph of G obtained by removing this edge from G . We first claim that x_{i1} is regular on S/P_G . Indeed, we have $(P_G, x_{i1}) = (P_{G'}, x_{11}x_{ii}, x_{i1})$. The height of the ideal $(P_{G'}, x_{11}x_{ii}, x_{i1})$ may be obtained by computing the height of its initial ideal with respect to \prec , which is $|E(G')| + 2 = |E(G)| + 1 = \text{height}(P_G) + 1$. Consequently, $\text{height}(P_G, x_{i1}) = \text{height}(P_G) + 1$, which shows that x_{i1} is regular on S/P_G since P_G is a complete intersection.

Now, the claim of the proposition follows if we show that $(S/P_G)_{x_{i1}}$ is a domain. We have $(S/P_G)_{x_{i1}} \cong (S'/P_{G'})[x_{i1}^{-1}]$, where S' is the polynomial ring in the variables $\{x_{ij} : 1 \leq i, j \leq n\} \setminus \{x_{1i}, x_{i1}\}$. Therefore, the proof is finished by applying induction on the number of edges of G . \square

Corollary 1.4. Let G be a simple graph on $[n]$. Then the ring $R_G = S/P_G$ is a normal domain.

Proof. Since $\text{in}_{\prec}(P_G)$ is a squarefree monomial ideal, the normality follows by applying a well known criterion of Sturmfels [11, Chapter 13]. \square

2. THE DIVISOR CLASS GROUP

Let G be a simple graph on $[n]$ and $R_G = S/P_G$. In the sequel, we are going to determine the divisor class group of R_G . We proceed as in the case of classical determinantal rings, see [12], [2]. Another useful reference on computing class groups of toric varieties is [3, Chapter 4].

We first choose an element $y \in R_G$ such that $(R_G)_y$ is a factorial ring. Then, by Nagata's Theorem [5, Corollary 7.2], we deduce that the divisor class group $\text{Cl}(R_G)$ is generated by the classes of the minimal prime ideals of y .

For a vertex i of G , we denote by $G \setminus \{i\}$ the subgraph of G obtained by removing the vertex i together with all the edges which are incident to i . For the next lemma we need some notation. For $i \in V(G)$ we denote by $N(i)$ the set of all the neighbors of i , that is, $N(i) = \{a \in V(G) : \{a, i\} \in E(G)\}$, and for each $a \in N(i)$, we set $E_a^i = \{x_{ai}, x_{ia}\}$.

Lemma 2.1. *Let $\{i, j\}$ with $i < j$ be an edge of G . Then:*

- (a) (P_G, x_{ji}) is an unmixed radical ideal with $\text{height}(P_G, x_{ji}) = \text{height } P_G + 1$.
- (b) The set of the minimal primes of (P_G, x_{ji}) is $\mathcal{C}_1(i, j) \cup \mathcal{C}_2(i, j)$, where

$$\mathcal{C}_1(i, j) = \{(P_{G \setminus \{i\}}, x_{ii}, x_{ji}, T)\}$$

where T is any set of variables with $T \subset \bigcup_{a \in N(i) \setminus \{j\}} E_a^i$ and $|T \cap E_a^i| = 1$ for all $a \in N(i) \setminus \{j\}$, and

$$\mathcal{C}_2(i, j) = \{(P_{G \setminus \{j\}}, x_{jj}, x_{ji}, U)\}$$

where U is any set of variables with $U \subset \bigcup_{b \in N(j) \setminus \{i\}} E_b^j$ and $|U \cap E_b^j| = 1$ for all $b \in N(j) \setminus \{i\}$.

Proof. We have $(P_G, x_{ji}) = (P_{G'}, x_{ii}x_{jj}, x_{ji})$, where G' is the subgraph of G obtained by removing the edge $\{i, j\}$. Since $\text{in}_<(P_{G'}, x_{ii}x_{jj}, x_{ji})$ is generated by a regular sequence of squarefree monomials of length $|E(G')| + 2 = |E(G)| + 1$, we get (a).

(b) Obviously, by a height argument, the ideals of the two classes are minimal primes of (P_G, x_{ji}) . Indeed, for example, for $i \in V(G)$ and any set T which defines an ideal of the set $\mathcal{C}_1(i, j)$, we have $\text{height}(P_{G \setminus \{i\}}, x_{ii}, x_{ji}, T) = \text{height}(P_{G \setminus \{i\}}) + |N(i)| + 1 = \text{height } P_G + 1$.

Let Q be a minimal prime of (P_G, x_{ji}) . As $x_{ji} \in Q$, we also have $x_{ii}x_{jj} \in Q$, hence $x_{ii} \in Q$ or $x_{jj} \in Q$. Let, for instance, $x_{ii} \in Q$. Then

$$Q \supset (P_G, x_{ii}, x_{ji}) = (P_{G \setminus \{i\}}, x_{ii}, x_{ji}, \{x_{ia}x_{ai} \mid a \in N(i) \setminus \{j\}\}).$$

Now, one easily sees that Q contains one of the ideals of the class $\mathcal{C}_1(i, j)$, and since $\text{height } Q = \text{height } P_G + 1$, Q must be equal to one of the ideals of the set $\mathcal{C}_1(i, j)$. \square

Let $y = \prod_{\substack{\{i, j\} \in E(G) \\ i < j}} \bar{x}_{ji} \in S/P_G$, and let $\bar{Q} = Q/P_G \subset S/P_G$ be a minimal prime of y . Then Q is a minimal prime of (P_G, x_{ji}) for some $\{i, j\} \in E(G), i < j$. Thus

Q belongs either to $\mathcal{C}_1(i, j)$ or to $\mathcal{C}_2(i, j)$. Therefore, the set $\text{Min}(y)$ of the minimal primes of (y) consists of all ideals $\overline{Q} \subset S/P_G$, where $Q \in \bigcup_{\{i,j\} \in E(G)} \mathcal{C}_1(i, j) \cup \mathcal{C}_2(i, j)$.

In order to determine the cardinality of the set $\bigcup_{\{i,j\} \in E(G)} \mathcal{C}_1(i, j) \cup \mathcal{C}_2(i, j)$, we observe that it is enough to count how many prime ideals Q contain x_{ii} for each $i \in V(G)$. But this is easy, since such an ideal Q is determined by a set $T \subset \bigcup_{a \in N(i)} E_a^i$ with $|T \cap E_a^i| = 1$ for all $a \in N(i)$ and with the property that for at least one variable $x_{cd} \in T$ we have $c > d$. Therefore, there are $2^{\deg i} - 1$ minimal primes of y which contain \overline{x}_{ii} . Consequently, we have

$$|\text{Min}(y)| = \sum_{i \in V(G)} (2^{\deg i} - 1) = \sum_{i \in V(G)} 2^{\deg i} - n.$$

Theorem 2.2. *The class group $\text{Cl}(R_G)$ is free of rank $\sum_{i \in V(G)} 2^{\deg i} - n - |E(G)|$.*

Proof. We first notice that in $(R_G)_y$ we have $\overline{x}_{ij} = \overline{x}_{ii}\overline{x}_{jj}\overline{x}_{ji}^{-1}$ for any edge $\{i, j\}$ of G with $i < j$. Therefore,

$$(R_G)_y \cong K[\{x_{ij} : i, j \in [n]\} \setminus \{x_{ij} : \{i, j\} \in E(G), i < j\}]_z,$$

where $z = \prod_{\substack{\{i,j\} \in E(G) \\ i < j}} x_{ji}$, which shows that $(R_G)_y$ is a factorial ring. By Nagata's

Theorem, it follows that the class group $\text{Cl}(R_G)$ is generated by the classes of the minimal primes of y . By Lemma 2.1, we get the following relations in $\text{Cl}(R_G)$:

$$(1) \quad \sum_{p \in \text{Min}(\overline{x}_{ji})} \text{cl}(p) = 0.$$

We show that all the relations between the classes of the minimal primes of y are linear combinations of the relations (1). Indeed, suppose that $\sum_{q \in \text{Min}(y)} m_q \text{cl}(q) = 0$ for some integers m_q .

This implies that $\sum_{q \in \text{Min}(y)} m_q \text{div}(q) = \text{div}(g)$, where $\text{div}(g)$ is a principal divisor in R_G . Since, $\text{div}(q)$ are in the kernel of the homomorphism $\text{Div}(R_G) \rightarrow \text{Div}((R_G)_y)$, it follows that g is a unit of $(R_G)_y$, hence $g = \lambda \prod_{\{i,j\} \in E(G)} \overline{x}_{ji}^{n_{ji}}$, for some integers n_{ji} and $\lambda \in K \setminus \{0\}$. Hence, we get $\sum_{q \in \text{Min}(y)} m_q \text{div}(q) = \sum_{\{i,j\} \in E(G)} n_{ij} \text{div}(\overline{x}_{ji}) = \sum_{\{i,j\} \in E(G)} n_{ij} (\sum_{p \in \text{Min}(\overline{x}_{ji})} \text{div}(p))$, thus $\sum_{q \in \text{Min}(y)} m_q \text{cl}(q)$ is a combination of relations of type (1) with coefficients n_{ij} . Then, by using the relations (1) for each $\{i, j\} \in E(G)$, we may express one class $\text{cl}(p)$ where $p \in \text{Min}(\overline{x}_{ji})$ as a combination of the others, and we get the statement of the theorem. \square

Now, we would like to answer the following question. Given a connected graph G with m edges, which are the bounds for the rank of the group $\text{Cl}(R_G)$? The answer is given in the following

Proposition 2.3. *Let G be a connected graph with m edges and let $R_G = S/P_G$, where P_G is the binomial ideal associated with G . Then*

$$2m - 1 \leq \text{rank Cl}(R_G) \leq 2^m - 1.$$

Moreover, $\text{rank Cl}(R_G) = 2m - 1$ if and only if G is the line graph and $\text{rank Cl}(R_G) = 2^m - 1$ if and only if G is the star graph.

Proof. We show, by induction on m , that if G is an arbitrary graph with m edges, then $\text{rank Cl}(R_G) \geq 2m - 1$. For $m = 1$, the claim is obvious. Let $m > 1$ and G a graph with m edges. We remove one edge $\{i, j\}$ of G and let G' be the new graph. It is clear that at least one of the vertices i and j has degree ≥ 2 , since G is connected. Then,

$$\begin{aligned} \text{rank Cl}(R_G) &= \sum_{a \in V(G)} (2^{\deg a} - 1) - m \\ &= \left[\sum_{\substack{a \in V(G), \\ a \neq i, j}} (2^{\deg a} - 1) + (2^{\deg i-1} - 1) + (2^{\deg j-1} - 1) - (m - 1) \right] + 2^{\deg i-1} + 2^{\deg j-1} - 1 \\ &= \text{rank Cl}(R_{G'}) + 2^{\deg i-1} + 2^{\deg j-1} - 1 \geq 2m - 3 + 2 = 2m - 1. \end{aligned}$$

We show that the only graph with m edges for which $\text{rank Cl}(R_G) = 2m - 1$ is the line graph with $m + 1$ vertices, by induction on m . The step $m = 1$ is clear. Let now $m > 1$ and assume that $\text{rank Cl}(R_G) = 2m - 1$. Then, by the above inequalities we get

$$2m - 1 = \text{rank Cl}(R_{G'}) + 2^{\deg i-1} + 2^{\deg j-1} - 1 \geq 2m - 3 + 2 = 2m - 1.$$

Therefore, we must have $\text{rank Cl}(R_{G'}) = 2m - 3$, thus G' is a line, by induction, and one of the vertices i, j has degree 1 and the other one has degree 2, which yields the desired conclusion.

For proving the inequality $\text{rank Cl}(R_G) \leq 2^m - 1$, we proceed by induction on m . More precisely, we first show that if G has no vertex of degree m , then $\text{rank Cl}(R_G) < 2^m - 1$. As in the first part of the proof, we remove an edge of G , let us say $\{i, j\}$. Then

$$\begin{aligned} \text{rank Cl}(R_G) &= \text{rank Cl}(R_{G'}) + 2^{\deg i-1} + 2^{\deg j-1} - 1 \\ &\leq 2^{m-1} - 1 + 2^{\deg i-1} + 2^{\deg j-1} - 1 < 2^m - 1, \end{aligned}$$

since G has no vertex of degree m . Indeed, we get the last inequality as follows

$$2^{m-1} + 2^{\deg i-1} + 2^{\deg j-1} - 2 \leq 2^{m-1} + 2 \cdot 2^{m-2} - 2 = 2^m - 2 < 2^m - 1.$$

The upper bound for $\text{rank Cl}(R_G)$ is clearly reached by the star graph, that is the graph with the edges $\{1, 2\}, \{1, 3\}, \dots, \{1, m\}, \{1, m + 1\}$, which is the only one which has a vertex of degree m . \square

Remark 2.4. One may easily see that, in general, not every integer between $2m - 1$ and $2^m - 1$ can be the rank of $\text{Cl}(R_G)$ for some connected graph G with m edges. For instance, for $m = 4$, the possible ranks of $\text{Cl}(R_G)$ are 7, 8, 9, 10 and 15.

Remark 2.5. Note that if G is the cycle with m edges, then $\text{rank Cl}(R_G) = 2m$. Therefore, for each positive integer n , one may find a graph G such that $\text{Cl}(R_G)$ is a free group of rank $\text{rank Cl}(R_G) = n$. Indeed, if $n = 2m - 1$, we may take G to be the line graph with m edges, and if $n = 2m$, we may take the cycle with m edges.

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, OVIDIUS UNIVERSITY, BD. MAMAIA
124, 900527 CONSTANTA, ROMANIA

E-mail address: `vivian@univ-ovidius.ro`

ABDUS SALAM SCHOOL OF MATHEMATICAL SCIENCES, GC UNIVERSITY, LAHORE. 68-B,
NEW MUSLIM TOWN, LAHORE 54600, PAKISTAN

E-mail address: `ayesqi@gmail.com`